Computation of the expected value of a function of a chi-distributed random variable

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Introduction

Consider the problem of numerically evaluating the expected value of a smooth bounded function of a chi-distributed random variable, divided by the square root of the number of degrees of freedom.

We use a transformation put forward by Mori, followed by the application of the trapezoidal rule. This rule has the remarkable property that, for suitable integrands, it is **exponentially convergent**. The application of the trapezoidal rule requires the approximation of an infinite sum by a finite sum. We provide a new easily computed upper bound on the error of this approximation.

Description of the Integrand of interest

Consider

$$\int_0^\infty a(x)\, f_\nu(x)\, dx, \tag{1}$$

where a is a smooth bounded real-valued function, ν is a positive integer and f_ν is the pdf of a random variable with the same distribution as $R/\nu^{1/2}$, where R has a χ_ν distribution.

Note that (1) $= E\big(a(R/\nu^{1/2})\big)$, which is the expected value of a smooth bounded function of $R/\nu^{1/2}$.

A better method for the evaluation of an integral of the form (1) is an appropriate transformation of the variable of integration, followed by application of the trapezoidal rule over the real line.

The trapezoidal rule has the remarkable property that, for suitable integrands, it is **exponentially convergent** (Trefethen & Weideman (2014)).

Trapezoidal rule found using the Fourier transform of the integrand

Suppose that we wish to evaluate

$$\int_{-\infty}^\infty g(y)\, dy, \tag{2}$$

where g is a real-valued absolutely integrable function. Let G denote that Fourier transform of g. This transform is defined by

$$G(\omega) = \int_{-\infty}^\infty g(y) \exp(-i\,\omega\, y)\, dy,$$

where $i = \sqrt{-1}$ and the angular frequency $\omega \in \mathbb{R}$. Since g is real-valued, $G(\omega)$ is an even function of ω. It follows from the Poisson summation formula that

$$\left| h \sum_{j=-\infty}^{\infty} g(jh+\delta) - \int_{-\infty}^\infty g(y)\, dy \right| \le 2 \sum_{j=1}^{\infty} \left| G\left(\frac{2\pi j}{h}\right)\right|,$$

for all $\delta \in [0, h)$. The left-hand side is the magnitude of the *discretization error*. This magnitude is small when $|G(\omega)|$ decays rapidly as $\omega \to \infty$ and h is sufficiently small.

We approximate

$$h \sum_{j=-\infty}^{\infty} g(jh+\delta) \quad \text{by} \quad h \sum_{j=M}^{N} g(jh+\delta) \tag{3}$$

for appropriately chosen integers M and N ($M < N$). The "trapezoidal rule" approximation to (2) is (3).

Transformation followed by the trapezoidal rule

To evaluate (1), we first apply the transformation,

$$x(y) = \exp\left(\frac{1}{2} y - e^{-y}\right)$$

so that

$$\int_0^\infty a(x)\, f_\nu(x)\, dx = \int_{-\infty}^\infty a(x(y))\, \psi_\nu(y)\, dy, \tag{4}$$

where

$$\psi_\nu(y) = f_\nu(x(y)) \frac{dx(y)}{dy}.$$

We will approximate (4) by

$$h \sum_{j=0}^{n-1} a\big(x(y_\ell + hj)\big)\, \psi_\nu(y_\ell + hj), \tag{5}$$

where n = number of evaluations, h = step length and the first evaluation of this integrand is at y_ℓ. Let $d = (n-1)h$.

Lemma Suppose that $y_\ell < y_\nu^*$ and that $y_\ell + d > y_\nu^*$. Then, when we approximate (4) by (5), the trimming error is bounded above by $u_\nu(y_\ell, d)$, where

$$u_\nu(y, d) = Q_\nu\Big(\nu\, x^2(y)\Big) + 1 - Q_\nu\Big(\nu\, x^2(y+d)\Big)$$

and Q_ν denotes the χ^2_ν cdf.

A simple procedure for evaluating the integral (4)

Step 1: Suppose that $\epsilon > 0$ is given. Choose d such that

$$\min_y u_\nu\big(y, d\big) = 10^{-3}\epsilon.$$

Choose y_ℓ to be the value of y minimizing $u\big(y, d\big)$. We have chosen the initial value of n to be 5. Proceed to the next step.

Step 2: For given (n, h, y_ℓ), evaluate the approximation (5) and store the result. Using the stored values of the approximations decide whether or not to stop the procedure. Proceed to the next step.

Step 3: Halve h and go back to the previous step

Results

Table 1: The approximation error for the trapezoidal rule for $\epsilon = 10^{-17}$. Here stopped after the computation of the approximation for $n = 65$ for $\nu = 1$ and $n = 33$ for others.

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$
$\alpha=0.10$	-1.11×10^{-16}	-2.22×10^{-16}	0	-2.22×10^{-16}	-1.11×10^{-16}
$\alpha=0.05$	9.66×10^{-15}	2.10×10^{-14}	-1.11×10^{-16}	-1.11×10^{-16}	-1.11×10^{-16}
$\alpha=0.02$	-1.23×10^{-12}	5.82×10^{-11}	9.99×10^{-16}	-1.11×10^{-16}	0

	$\nu=10$	$\nu=100$	$\nu=1000$
$\alpha=0.10$	2.00×10^{-15}	2.78×10^{-14}	-2.37×10^{-13}
$\alpha=0.05$	2.11×10^{-15}	2.94×10^{-14}	-2.49×10^{-13}
$\alpha=0.02$	2.22×10^{-15}	3.03×10^{-14}	-2.57×10^{-13}

Table 2: The approximation error for Generalized Gauss Laguerre quadrature. The number of nodes is 65 for $\nu = 1$ and 33 for others.

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$
$\alpha=0.10$	1.44×10^{-2}	1.32×10^{-3}	1.63×10^{-4}	2.86×10^{-5}	6.08×10^{-6}
$\alpha=0.05$	3.25×10^{-2}	2.04×10^{-3}	2.26×10^{-4}	3.77×10^{-5}	7.84×10^{-6}
$\alpha=0.02$	2.00×10^{-2}	4.12×10^{-3}	3.39×10^{-4}	5.24×10^{-5}	1.05×10^{-5}

	$\nu=6$	$\nu=10$	$\nu=100$	$\nu=300$
$\alpha=0.10$	1.48×10^{-6}	1.23×10^{-8}	-2.00×10^{-14}	-6.22×10^{-15}
$\alpha=0.05$	1.88×10^{-6}	1.52×10^{-8}	-2.11×10^{-14}	-6.21×10^{-15}
$\alpha=0.02$	2.46×10^{-6}	1.91×10^{-8}	-2.18×10^{-14}	-6.43×10^{-15}

Table 3: The approximation error for Gauss Legendre quadrature. The number of nodes is 65 for $\nu = 1$ and 33 for others.

	$\nu=1$	$\nu=2$	$\nu=3$	$\nu=4$	$\nu=5$
$\alpha=0.10$	7.77×10^{-16}	6.39×10^{-6}	1.52×10^{-5}	2.07×10^{-5}	2.37×10^{-5}
$\alpha=0.05$	8.88×10^{-16}	9.43×10^{-6}	2.06×10^{-5}	2.70×10^{-5}	2.96×10^{-5}
$\alpha=0.02$	8.88×10^{-16}	1.53×10^{-5}	2.94×10^{-5}	3.58×10^{-5}	3.75×10^{-5}

	$\nu=6$	$\nu=10$	$\nu=100$	$\nu=1000$
$\alpha=0.10$	2.47×10^{-5}	2.30×10^{-5}	4.01×10^{-6}	4.23×10^{-7}
$\alpha=0.05$	3.02×10^{-5}	2.64×10^{-5}	4.06×10^{-6}	4.23×10^{-7}
$\alpha=0.02$	3.68×10^{-5}	2.90×10^{-5}	3.36×10^{-6}	3.33×10^{-7}

Conclusion

The method of using the transformation followed by the trapezoidal rule performs more or less equally well for all of the values of ν considered. The Generalized Gauss Laguerre quadrature performs worst for $\nu = 1$, and has performance that improves with increasing ν. The inverse cdf method performs best for $\nu = 1$, and has performance that decreases as ν increases through the values $2, 3, 4, 5, 6$ and 10.

References

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